Math-601D-201: Lecture 19.
Pseudo-convex domains

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Pseudo-convex domains

$\Omega \subset \mathbb{C}^n$ connected open set.

Definition

Ω is a pseudo-convex domain iff for any compact set $K \subset \Omega$, the set

$$\hat{K}_{PSH(\Omega)} = \bigcap_{u \in PSH(\Omega)} \{z \in \Omega,\ u(z) \leq \sup_K u\}$$

is compact in Ω.

Observation

$$\hat{K}_{PSH(\Omega)} \subset \hat{K}_{\mathcal{O}(\Omega)} \equiv \hat{K}_\Omega$$

hence any domain of holomorphy is a pseudo-convex domain

Characterization of pseudo-convex domains

$\Omega \subset \mathbb{C}^n$ connected open set.

Theorem

The following are equivalent:

1. Ω *is pseudo-convex;*
2. $-\log dist(\cdot, \partial\Omega)$ *is psh;*
3. *there exists* $u \in PSH(\Omega) \cap \mathcal{C}^0(\Omega)$ *such that for all* $c \in \mathbb{R}$

$$\Omega_c(u) = \{u < c\} \text{ is relatively compact in } \Omega$$

Geometric characterization of pseudo-convex domains

$\Omega \subset \mathbb{C}^n$ connected open set.

Definition

Ω is geometrically pseudo-convex iff for any continuous family of holomorphic disks $F\colon [0,1] \times \bar{\mathbb{D}} \to \mathbb{C}^n$ such that

- *$z \mapsto F(t,z)$ is holomorphic;*
- *$F(t,z) \subset \Omega$ if $t < 1$;*
- *$F(1,z) \subset \Omega$ if $|z| = 1$*

then $F(1,z) \in \Omega$ for all $|z| \leq 1$.

Theorem

A domain is pseudo-convex iff it is geometrically pseudo-convex.

Being pseudo-convex is a local property

$\Omega \subset \mathbb{C}^n$ connected open set.

Theorem

Ω is pseudo-convex iff for any $p \in \partial\Omega$, one can find an open neighborhood $\omega \ni p$ such that $\Omega \cap \omega$ is pseudo-convex.